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LETTER TO THE EDITOR

Nonlinear dynamics from the Wilson Lagrangian

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Abstract. A nonlinear Hamiltonian dynamics is derived from the Wilson action in lattice gauge theory. Let $\mathcal{D}$ be a linear space of lattice Dirac operators $D(a)$ defined by some lattice gauge field a. We consider the Lagrangian $D \mapsto \mathrm{tr}((D(a) + \mathrm{i}m)^4)$ on $\mathcal{D}$, where $m \in \mathbb{C}$ is a mass parameter. Critical points of this functional are given by solutions of a nonlinear discrete wave equation which describe the time evolution of the gauge fields a. In the simplest case, the dynamical system is a cubic Henon map. In general, it is a symplectic coupled map lattice. We prove the existence of non-trivial critical points in two examples.

1. The problem

Let $\mathcal{X}$ be an operator algebra with finite trace tr. We introduce the problem of finding critical points of the functional

$$\mathcal{L}_m : D \mapsto \mathrm{tr}((D + \mathrm{i}m)^4) \tag{1}$$

on some linear subspace $\mathcal{D}$ of $\mathcal{X}$, where m is a complex mass parameter. If $\mathcal{D}$ is formed by discrete Dirac operators $D = \sum_j a_j \tau_j + (a_j \tau_j)^*$, where the a_i are in a subalgebra $\mathcal{A} \subset \mathcal{X}$ and τ_j are automorphisms in $\mathcal{A}$, this functional is an averaged Wilson action of the lattice gauge field a_j. We demonstrate here that critical points of $\mathcal{L}_m$ define a nonlinear dynamical system and look at examples. In the simplest case, if all the a_j are invariant under space translations, the time evolution is given by a cubic Henon twist map in the plane. In general, the dynamics is an infinite dimensional nonlinear discrete reversible wave equation. These discrete partial difference equations are generalizations of classical coupled map lattices [13, 10, 5]. Hamiltonian reversible coupled map lattices appeared in [12, 15]. Non-invertible coupled map lattices in connection with field theory were treated in [3]. Here, we have a both space and time-discrete wave equation, time is physical time and we work in an ergodic set-up.

2. The motivation

Non-relativistic quantum dynamics deals with the Schrödinger dynamical system $\mathrm{i}\hbar\dot{\psi} = L\psi$ in a Hilbert space $\mathcal{H}$. If L is a bounded operator, the discrete time version

$$\mathrm{i}\frac{\hbar}{2\epsilon}[\psi(t+\epsilon) - \psi(t-\epsilon)] = L\psi(t) \tag{2}$$

defines a unitary evolution and is useful for studying spectral measures of L ([19]). If the left-hand side of equation (2) is $\mathrm{i}a^{-1}(U - U^*)$ and $V = \mathrm{i}U$, then equation (2) is $V + V^* = aL$

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a discretization of a relativistic wave equation. Using the discrete operator $K = V + V^* - aL$ on spacetime, the evolution (2) is equivalent to $K\psi = 0$, so that a wave ψ is a critical point of the formal functional $\psi \mapsto (\psi, K\psi)$. In quantum field theory, the waves ψ become operators and contribute to the Hamiltonian. We assume that both the wave $\psi = D$ and the Hamiltonian $K = D^2 - m^2$ are in an C^* algebra $\mathcal{X}$ with finite trace tr. $\mathcal{X}$ is a not yet completed normed space with scalar product $(A, B) = \mathrm{tr}(A, B)$. We replace the functional $\psi \mapsto (\psi, K\psi)$ leading to a linear unitary quantum evolution (2) by the smooth bounded functional $D \mapsto (D, (D^2 - m^2)D) = \mathcal{L}_m - m^4$. It gives a nonlinear evolution on $\mathcal{D}$ and on its completion, the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$. While the functional (1) has on $\mathcal{X}$ only the trivial critical points $D = -\mathrm{i}m$, it becomes interesting when restricted to a linear space $\mathcal{D} = \{D = \sum_{j=1}^d a_j\tau_j + (a_j\tau_j)^*, a_j \in \mathcal{A}\} \subset \mathcal{X}$, where τ_j are fixed unitary elements in $\mathcal{X}$ defining automorphisms in $\mathcal{A}$ and $\mathcal{D}$ by $a \mapsto \tau_j a \tau_j^*$ and $D \mapsto D(T_j) = \tau_j D \tau_j^*$, respectively. Every $D \in \mathcal{D}$ defines a real-valued lattice gauge field on the Cayley graph of the group $\mathcal{Z}$ generated by the unitaries τ_j. At the bond connecting D with $D(T_j)$ is attached the field a_j. The value of $\mathcal{L}_m$ is a sum over all parallel transports of closed paths of length 4. If the gauge fields a_j are unitary and Ω is finite then $\mathcal{L}_m$ is up to a linear transformation exactly the Wilson action. In [17] we considered the variational problem

$$D \mapsto \mathrm{Det}(D + m) = \exp(\mathrm{tr}(\log|(D + m)|)) \tag{3}$$

which is not smooth in infinite dimensions. Minimizers of (3) exist for all $m \in \mathbb{C}$ in a statistical mechanical set-up, where D is defined by a translation-invariant measure on the set of unitary gauge field configurations. The functional (3) defines no dynamics. $\mathrm{tr}(D^2)$ gives no interesting evolution. The expansion $m^{-1}\,\mathrm{Det}(D + m) = 1 - \mathrm{tr}(D^2)m^{-2}/2 + (\mathrm{tr}(D^2)^2 - 2\,\mathrm{tr}(D^4))m^{-4}/8 + \mathrm{O}(m^{-6})$ makes (1) the simplest substitute for the determinant (3). Higher-order polynomial actions do not lead to wave equations: critical points would lead to unphysical higher-order PDEs and so violate the primitive causality axiom [11]. Another motivation is of course that the Wilson action is the Hamiltonian in lattice gauge theory. It is fundamental because it becomes in the continuum limit the Maxwell–Yang–Mills functional. A more general problem which we do not address in this letter is to allow $\mathcal{D}$ to be a manifold in $\mathcal{X}$ like for example $\mathcal{D} = \{D = \sum_j a_j\tau_j + (a_j\tau_j)^*,\ a_j^{-1} = a_j^* \in \mathcal{A}\}$ leading to a variational problem with constraints.

3. The functional

Let $\mathcal{R}$ be any group acting as automorphisms on a probability space (Ω, ν). Let $\mathcal{Z}$ be a discrete subgroup of $\mathcal{R}$ with generators $T_k : \Omega \to \Omega$ and automorphisms $\tau_k : a \mapsto a(T_k)$ on $\mathcal{A} = L^\infty(\Omega, \nu)$. Let $\mathcal{X}$ be the crossed product of $\mathcal{A}$ with the $\mathcal{Z}$ action. We call the elements $D = \sum_k a_k\tau_k + (a_k\tau_k)^* \in \mathcal{X}$ discrete Dirac operators. In the discrete case, matrix-valued coefficients $\gamma_k a_k$ with Dirac matrices γ_k are not necessary because the Clifford relations can be achieved on a doubled lattice, which leads for dim $= 4$ to operators on spinors with 16 components. Let $\mathcal{D}$ be a linear space of such operators. We look at the problem of finding critical points of the action (1) on $\mathcal{D}$, where $m \in \mathbb{C}$ is a parameter. The choice $\mathcal{R} = \mathcal{P} \times \mathcal{G}$, where $\mathcal{P}$ is the Poincaré group and where $\mathcal{G}$ is a product of compact Lie groups, is physically motivated. $\mathcal{Z} \subset \mathcal{R}$ is the product of a discrete lattice $\mathbb{Z}^4$ in the translation group generated by $\tau_0, \ldots, \tau_3$ of $\mathcal{P}$ and a countable group in $\mathcal{G}$ generated by σ_j. Every $D = \sum_l a_l\tau_l + (a_l\tau_l)^* = \sum_{j=0}^3 a_j\tau_j + (a_j\tau_j)^* + \sum_k b_k\sigma_k + (b_k\sigma_k)^*$, is the sum of a kinematic part and a part responsible for internal degrees of freedom. If a_j and b_k commute with involutions σ_k, we write $D = \sum_j a_j\tau_j + (a_j\tau_j)^* + \sum_k 2b_k\sigma_k$.

1. Critical points. Let π be the projection from $\mathcal{X}$ to $\mathcal{D}$ defined by $(\cdot,\cdot)$. A functional $D \mapsto \mathcal{L}_f(D) = \mathrm{tr}(f(D))$ on $\mathcal{D}$ has the Fréchet derivative $d\mathcal{L}(D)(U) = \mathrm{tr}(U\pi f'(D))$ and so the functional derivative $\delta\mathcal{L}_f = \pi f'(D)$. In particular, the Euler equations of the functional $\mathcal{L}_m = \mathrm{tr}((D+im)^4)$ are $\pi(D+\mathrm{i}m)^3 = 0$. Explicitly,

$$0 = -3m^2 a_k + a_k\left(\sum_{j\neq k} a_j^2(T_k) + a_j^2(T_kT_j^{-1}\right) + a_j^2 + a_j^2(T_j^{-1}))$$

$$+a_k(a_k^2(T_k) + a_k(T_k^{-1})^2 + a_k^2)$$

$$+\sum_{j\neq k} a_k(T_j)a_ja_j(T_k) + a_k(T_j^{-1})a_j(T_j^{-1})a_j(T_j^{-1}T_k).$$

Let $\mathcal{D}_0 = \{D \in \mathcal{D} \mid D(T_0) = D\}$. Because $a_k(T_0)$ and $a_l(T_0)$ do not occur simultaneously in one equation for $k \neq l, k \neq 0, l \neq 0$, one can solve for $a_k(T_0)$ and then for $a_0(T_0)$. This gives a symplectic map S on $\mathcal{D}_0 \times \mathcal{D}_0$. We will write it down only in the examples. Given an S-invariant measure μ on $\mathcal{D}_0 \times \mathcal{D}_0$, which is T_i-invariant for $i \geqslant 1$. Define a new probability space $\Omega = \mathcal{D}_0 \times \mathcal{D}_0$ and $D((\omega_1,\omega_2)) = \omega_1$. Ω carries an additional action S replacing T_0 and commuting with T_j. Denoting again by $\mathcal{X}$ the crossed product of $\mathcal{A} = L^\infty(\Omega,\mu)$ with this new $\mathcal{Z}$ action and by $\mathcal{D}$ the corresponding subspace, then D is a critical point of $\mathcal{L}_m$. Having in mind the Wightman axioms [9], we choose μ ergodic. The space- and time-invariant field D is then essentially unique.

2. The Hessian. The map $D \mapsto \pi D^n$ on $\mathcal{D}$ has the linearization $U \mapsto \pi\sum_{k=0}^{n-1} D^k U D^{n-1-k}$. The Hessian $\delta^2\mathcal{L}(D)$ at a critical point D is therefore the linear map on $\mathcal{D}$

$$U \mapsto LU = \pi[(D+\mathrm{i}m)U(D+\mathrm{i}m) + U(D+\mathrm{i}m)^2 + (D+\mathrm{i}m)^2U].$$

For $U = u_k\tau_k + u_k\tau_k^* \in \mathcal{D}$, we obtain $LU = v_k\tau_k + (v_k\tau_k)^*$ with

$$v_k = -3m^2u_k + \sum_{j\neq k}(a_j^2 + a_j^2(T_kT_j^{-1}) + a_j^2(T_k) + a_j^2(T_j^{-1}))u_k$$

$$+(a_k^2(T_k) + a_k^2(T_k^{-1}) + 3a_k^2)u_k$$

$$+\sum_j a_ja_j(T_k)u_k(T_j) + a_j(T_j^{-1})a_j(T_kT_j^{-1})u_k(T_j^{-1}).$$

If L is invertible, the critical point is structurably stable with respect to changes in m. This fact can be useful for constructing critical points perturbatively, as we will see in an example.

3. Gauge invariance. For $g \in \mathcal{A}$, define the gauge transformation $D \mapsto gDg^{-1}$ on $\mathcal{D}$. The lattice gauge field a defined by D transforms like $a_k \mapsto ga_kg(T_k^{-1})^{-1}$. The trace property $\mathrm{tr}(gf(D)g^{-1}) = \mathrm{tr}(f(D))$ implies that $\mathcal{L}_m$ is gauge invariant. In particular, if D is a critical point, then gDg^{-1} is a critical point too. Gauge transformations are unitary with respect to the scalar product $\mathrm{tr}(\cdot,\cdot)$ on $\mathcal{D}$.

4. Fields. Each operator D defines a lattice gauge field or discrete 1-form $a = \sum_j a_j \tau_j$. If $\mathcal{Z}$ is Abelian, define the field $F = da = \sum_{i<j} F_{ij}\tau_i\tau_j$, where $F_{ij} = a_j^{-1} a_i(T_j)^{-1} a_j(T_i) a_i$ is the result of the parallel transport around the ij plaquette. The field F is gauge invariant. Since the a_i are in an Abelian algebra $\mathcal{A}$, the 3-form dF satisfies

$$dF_{ijk} = F_{ij}F_{ij}(T_k)^{-1}F_{jk}F_{jk}(T_i)^{-1}F_{ki}F_{ki}(T_j)^{-1} = 1$$

where 1 is the zero element in the gauge group. The Maxwell equation $dF = 1$ is a special case of the general fact $d \circ d = 0$ in ergodic group cohomology [7, 18].

5. Extension of the functional. Let $\mathcal{R}$ have the discrete topology and let $\mathcal{Y}$ be the crossed product of $\mathcal{A} = L^\infty(\Omega)$ with the $\mathcal{R}$ action. For $e = (e_1, e_2, \ldots,)$ with $e_i \in \mathcal{R}$ and $a_i \in \mathcal{A}$, consider $\mathcal{D}_e = \{D = \sum_j a_j\tau_{e_j} + (a_j\tau_{e_j})^*\} \subset \mathcal{Y}$. The group $\mathcal{R}$ acts on $\bigcup_e \mathcal{D}_e$ by

$$D(a, e) \mapsto D_r(a, e) = D(a(T^r), rer^{-1}).$$

All $D_r, r \in \mathcal{R}$ have the same density of states and $\mathcal{L}_m$ is invariant under the $\mathcal{R}$ action on $\bigcup_e \mathcal{D}_e$. Because $\mathrm{tr}(D^{(1)}D^{(2)}) = \mathrm{tr}(D_r^{(1)}D_r^{(2)})$, the $\mathcal{R}$ action on $\mathcal{D}$ defines a unitary representation of $\mathcal{R}$ on the completion $\mathcal{H}$ of the pre-Hilbert space $(\bigcup_i \mathcal{D}_e, \mathrm{tr}(\cdot,\cdot))$. If $\mathcal{R}$ is the Poincaré group $\mathcal{P}$, then $D(a, e) \mapsto D_r(a, e) = D(a(T^r), P^{-r}e)$, where $T^r = P^r + p^r$ is the decomposition into Lorentz transformation and translation.

6. Discrete spacetime and continuous symmetry. The functional $\mathcal{L}_m$ is $\mathcal{R}$-invariant in the algebra $\mathcal{Y}$ of all possible discrete operators. We presently do not consider variations in $\mathcal{Y}$ and fix the lattice. We can allow variations of the vector e defining the discrete subgroup $\mathcal{Z}$ if $\mathcal{R}$ acts by diffeomorphisms on a manifold M. The general variational problem in $\mathcal{Y}$ is Lorentz invariant. For discrete time quantum mechanics see [22]. Discreteness emerges naturally in the ergodic theory of $\mathcal{R}$ actions [6].

4. Examples

1. A one-dimensional example. Let $T = T_0, T_1 : \Omega \to \Omega$ be commuting measure-preserving transformations. Consider $\mathcal{D} = \{D = \tau + \tau^* + b\sigma \mid b = b(T_1) \in L^\infty(\Omega)\}$. The variational problem (1) is $\mathcal{L}_m : D \mapsto \mathrm{tr}((D + \mathrm{i}m)^4) = \mathrm{tr}(D^4) - 6m^2\,\mathrm{tr}(D^2) + m^4$ and has the critical points $D^3 - 3m^2 D = 0$. Finding a bounded measureable map b satisfying the Euler equations $b(T_0) + b(T_0^{-1}) + 4b - 3m^2 b + b^3 = 0$ is equivalent to find an invariant measure m of the cubic Henon map

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (-(4 - 3m^2)x - x^3 - y \\ x \end{pmatrix} \qquad (4)$$

where $x = b, y = b(T_0^{-1})$: the solution b is the first coordinate of the factor map $\Omega \to \mathbb{R}^2$ conjugating (Ω, T, ν) with $(\mathbb{R}^2, S, \mu)$ [16]. Another generating function for S is $-\frac{1}{2}(b(T_0) - b)^2 + V(b)$ with $V(b) = \frac{1}{4}b^4 + \frac{1}{2}(4 - 3m^2)b^2$. Such maps S are in general non-integrable [20] and the non-wandering set is compact [8]. Critical points can be constructed (i) using results in smooth ergodic theory or (ii) by KAM perturbation theory: (i) if there is a periodic orbit of odd period $p > 1$, then the topological entropy $h_{\mathrm{top}}(S)$ is positive [4]. Moreover, $h_{\mathrm{top}}(S)$ takes the maximal value log(3) if $|4 - 3m^2| > 3g^{2/3}$ due to an embedded horseshoe. [8]. $h_{\mathrm{top}}(S) > 0$ implies the existence of a compact invariant hyperbolic invariant set [14]. (ii) If $|4 - 3m^2| < 2$, then 0 is a linearly stable fixed point. It is stable for the generic set of m's with non-trivial Birkhoff normal form [21]. This assures the existence of invariant measures even absolutely continuous with respect to Lebesgue measure. The Hessian is the bounded random operator on $l^2(\mathbb{Z})$ $\delta^2\mathcal{L}_m(D) : u \mapsto u_{n+1} + u_{n-1} - 2u_n + V''(x_n)u_n$, where $x_n = b(T^n)$ is obtained from the function b defining the critical point D.

2. A higher-dimensional example. Given $\mathcal{D} = \{D = \tau + \tau^* + \alpha \sum_{j=1}^{d}(\tau_j + \tau_j^*) + b\sigma\}$. A critical point of $\mathcal{L}_m$ satisfies the Euler equations

$$b(T_0) + b(T_0^{-1}) + (4 - 3m^2)b + \alpha^2\Delta b + 4d\alpha^2 b + b^3 = 0 \tag{5}$$

where $\Delta b = \sum_{j=1}^{d} b(T_j) + b(T_j^{-1})$. Writing $x = b, y = b(T_0^{-1})$, we obtain the discrete Hamiltonian system

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (-\alpha^2(\Delta + 4d)x - (4 - 3m^2)x - x^3 - y \\ x \end{pmatrix} \tag{6}$$

where S defines an invertible *coupled map lattice*. Invertible coupled map lattices of similar type have been considered in [12]. For $\alpha = 0$, S is an array decoupled cubic Cremona maps. They are, for $\alpha > 0$, linked through a linear nearest-neighbour coupling. Equation (6) is a discrete version of the nonlinear ϕ^4 wave equation $(\Box + m^2)\phi = p(\phi)$, where $\Box = \partial_t^2 - \Delta$ and p is a cubic polynomial. If Ω is a finite set, S is a symplectic map $(x, y) \mapsto (f(x) - y, x)$ on $\mathbb{R}^{2|\Omega|}$. The linearization of S is conjugated to a decoupled system of two unitary Schrödinger evolutions if m is choosen so that the Hessian is not invertible. While the nonlinear Schrödinger equation is integrable by an infinite dimensional Siegel theorem [23], the nonlinear ϕ^4 wave analogue is not. An analogue fact holds here in the special case of functions which are constant in space: while a polynomial map $x \mapsto f(x)$ can be integrable near 0 by Siegel's theorem, the symplectic map $(x, y) \mapsto (f(x) - y, x)$ is non-integrable and only a deeper KAM argument can establish stability of an elliptic fixed point. In higher dimensions, a linearly stable fixed point 0 is in general unstable. We now prove for large m that there are aperiodic non-trivial solutions of the dynamical system (6) using an argument of Aubry [1, 2, 12, 16, 15]. Equation (5) is equivalent to $F(\epsilon, q) = 0$, where $q = b/m, \epsilon = m^{-3}$ and

$$F(\epsilon, q) = \epsilon[q(T_0) + q(T_0^{-1}) + \alpha^2\Delta q + (4d\alpha^2 + 4)q] - 3q + q^3.$$

For $\epsilon = 0$, solve $-3q + q^3 = 0$ by any function $q : \Omega \to \mathbb{R}$ taking values in $\{\pm\sqrt{3}\}$. The linear map $(\partial/\partial q)F(0, q)$ on $L^\infty(\Omega)$ is invertible. The implicit function theorem gives solutions q for small $\epsilon = m^{-2}$. This argument generalizes to find the critical points for large m for

$$\mathcal{D} = \left\{ D = \tau + \tau^* + \sum_{j=1}^{d} \alpha(\tau_j + \tau_j^*) + \sum_{j=1}^{n} b_j\sigma_j \right\}$$

where the critical points satisfy

$$b_i(T_0) + b_i(T_0^{-1}) + (4 - 3m^2)b_i + \alpha^2\Delta b + b_i^3 + 3b_i \sum_{k \neq i} b_k^2 = 0.$$

The corresponding symplectic map S has also the generating function

$$\sum_j \left(\frac{(b_j(T_0) - b_j)^2}{2} + \alpha^2 \sum_k \frac{(b_j(T_k) - b_j)^2}{2} + (2\alpha^2 + 6 - 3m^2)\frac{b_j^2}{2} + \frac{b_j^4}{4} + \frac{3}{2}\sum_k b_j^2 b_k^2 \right).$$

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